Quantum Fluctuations of the Center of Mass and Relative Parameters of Nonlinear Schrödinger Breathers

Oleksandr V. Marchukov,[1,2,*] Boris A. Malomed,[2,3] Vanja Dunjko,[4] Joanna Ruhl,[4] Maxim Olshanii,[4] Randall G. Hulet,[5] and Vladimir A. Yurovsky[6]

[1]*Institute for Applied Physics, Technical University of Darmstadt, 64289 Darmstadt, Germany*
[2]*Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, and Center for Light-Matter Interaction, Tel Aviv University, 6997801 Tel Aviv, Israel*
[3]*Instituto de Alta Investigación, Universidad de Tarapacá, Casilla 7D, Arica, Chile*
[4]*Department of Physics, University of Massachusetts Boston, Boston, Massachusetts 02125, USA*
[5]*Department of Physics and Astronomy, Rice University, Houston,Texas 77005, USA*
[6]*School of Chemistry, Tel Aviv University, 6997801 Tel Aviv, Israel*

(Received 4 November 2019; revised 6 March 2020; accepted 2 July 2020; published 28 July 2020)

We study quantum fluctuations of macroscopic parameters of a nonlinear Schrödinger breather—a nonlinear superposition of two solitons, which can be created by the application of a fourfold quench of the scattering length to the fundamental soliton in a self-attractive quasi-one-dimensional Bose gas. The fluctuations are analyzed in the framework of the Bogoliubov approach in the limit of a large number of atoms N, using two models of the vacuum state: white noise and correlated noise. The latter model, closer to the *ab initio* setting by construction, leads to a reasonable agreement, within 20% accuracy, with fluctuations of the relative velocity of constituent solitons obtained from the exact Bethe-ansatz results [Phys. Rev. Lett. **119**, 220401 (2017)] in the opposite low-N limit (for $N \leq 23$). We thus confirm, for macroscopic N, the breather dissociation time to be within the limits of current cold-atom experiments. Fluctuations of soliton masses, phases, and positions are also evaluated and may have experimental implications.

DOI: 10.1103/PhysRevLett.125.050405

Introduction.—The nonlinear Schrödinger equation (NLSE) plays a fundamental role in many areas of physics, from Langmuir waves in plasmas [1] to the propagation of optical signals in nonlinear waveguides [2–6]. A variant of the NLSE, in the form of the Gross-Pitaevskii equation (GPE), provides the mean-field (MF) theory for rarefied Bose-Einstein condensates (BECs). Experimentally, bright solitons predicted by the GPE with the self-attractive nonlinearity were observed in ultracold ^{7}Li [7–9] and ^{85}Rb [10,11] gases in the quasi-one-dimensional (1D) regime imposed by a cigar-shaped potential trap. Because the GPE-based MF approximation does not include quantum fluctuations, one needs to incorporate quantum many-body effects to achieve a more realistic description of the system. The simplest approach is to employ the linearization method first proposed by Bogoliubov [12] in the context of superfluid quantum liquids. For more than two decades, this method has been successfully used to describe excitations in BECs [13–16]. Another approach deals with the Lee-Huang-Yang (LHY) corrections [17] to the GPE induced by quantum fluctuations around the MF states [18,19]. The so improved GPEs produce stable 2D and 3D solitons (including ones with embedded vorticity [20,21]), which have been created in experiments with binary [22–25] and single-component dipolar [26,27] BECs.

The focusing nonlinearity in the NLSE corresponds to attractive interactions between atoms in BEC. The NLSE in 1D without external potentials belongs to a class of integrable systems [28–30], thus maintaining infinitely many dynamical invariants and infinitely many species of soliton solutions. The simplest one, the fundamental bright soliton, is a localized stationary mode which can move with an arbitrary velocity. The next-order solution, i.e., a two soliton, which is localized in space and oscillates in time, being commonly called a *breather*, can be found by means of an inverse-scattering transform [31]. This solution may be interpreted as a nonlinear bound state of two fundamental solitons with a 1 : 3 mass ratio and exactly zero binding energy [29,32]. The two-soliton breather can be created by a sudden quench of the interaction strength, namely, its fourfold increase, starting from a single fundamental soliton, as was predicted long ago in the analytical form [31], and recently demonstrated experimentally in BEC [33].

Quantum counterparts of solitons and breathers can be constructed as superpositions of Bethe ansatz (BA) eigenstates of the corresponding quantum problem [5] which recover MF properties in the limit of large number of atoms (N). While an experimental observation of the quantum behavior of the center-of-mass (COM) coordinate of a (macro or meso)-scopic soliton (e.g., effects analyzed in Refs. [34–36]) remains elusive, several groups have been making progress towards this goal [33,37]. Certain quantum features of NLSE breathers, such as correlations and squeezing [38–40], conservation laws [41], development of decoherence [42], and nonlocal correlations [43], have been analyzed and discussed. The non-MF breatherlike solutions were also considered in open Bose-Hubbard, sine-Gordon, and other models [44,45]. Note that in the semiclassical limit the instability of quantum breathers carries over into the MF regime that was explored for NLSE in various settings in Ref. [46].

At the MF level, the relative velocity of the fundamental solitons, whose bound state forms the breather, is identically equal to zero, regardless of how hot the COM state of the "mother" soliton was. Thus, if the breather spontaneously splits in free space into a pair of constituent fundamental solitons, intrinsic quantum fluctuations are expected to be the *only* cause of the fission (at the MF level, controllable splitting of the breather can be induced by a local linear or nonlinear repulsive potential [47]). This expectation suggests a way to observe the splitting as a direct manifestation of quantum fluctuations in a macroscopic object, which may take place under standard MF experimental conditions.

The Bogoliubov linearization method was first applied to fundamental solitons [48–50] in optical fibers. Later, Yeang [51] extended the analysis for the COM degree of freedom of a breather. The present work focuses on quantum fluctuations of breather's relative parameters. We deal with two models for the halo of quantum fluctuations around the MF states of the atomic BEC: conventional "white noise" [42,48,49] of vacuum fluctuations, and the most relevant scheme with correlated noise, assuming that the breather has been created from a fluctuating fundamental soliton, by means of the aforementioned factor-of-four quench, as schematically shown in Fig. 1. For a small number of atoms, up to $N = 23$, estimates for the relative velocity variance and splitting time were obtained in Ref. [52], using the exact many-body BA solution; however, available techniques do not make it possible to run experiments with such "tiny solitons." The present Letter extends the results for the experimentally relevant large values of N and provides variances of other breather parameters, which may also be observable.

The system.—We consider a gas of bosons with s-wave scattering length $a_{\rm sc} < 0$ in an elongated trap with transverse trapping frequency $\omega_\perp$ [7,8,53]. The scattering length can be tuned by a magnetic field, using the Feshbach resonance [54]. Atoms with kinetic energy $< \hbar\omega_\perp$ may be considered as 1D particles with the attractive zero-range interaction between them, of strength $-g = 2\hbar\omega_\perp a_{\rm sc}$ [55]. The 1D gas is described by the quantum (Heisenberg's) NLSE,

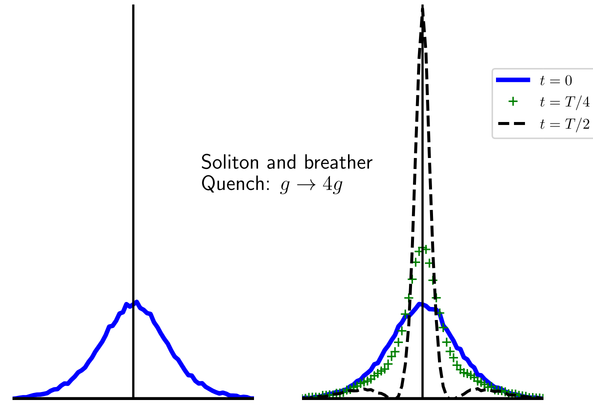

FIG. 1. A schematic representation of the fundamental "mother soliton", as the vacuum state including inherent correlated quantum noise (the left panel), transformed into the breather by means of the interaction quench (the right panel).

$$i\hbar\frac{\partial\hat{\Psi}(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\hat{\Psi}(x,t)}{\partial x^2} - g\hat{\Psi}^\dagger(x,t)\hat{\Psi}(x,t)\hat{\Psi}(x,t), \tag{1}$$

where m is the atomic mass. The creation and annihilation quantum-field operators, $\hat{\Psi}^\dagger$ and $\hat{\Psi}$, obey the standard bosonic commutation relations.

The Bogoliubov theory represents the quantum field as $\hat{\Psi}(x,t) = \sqrt{N}\Psi_0(x,t) + \delta\hat{\psi}(x,t)$, where the first MF term is a solution of classical NLSE representing the condensed part of the boson gas. Operator $\delta\hat{\psi}(x,t)$ represents quantum fluctuations, also obeying the standard bosonic commutation relations. The Bogoliubov method linearizes Eq. (1) with respect to $\delta\hat{\psi}$:

$$i\hbar\frac{\partial\delta\hat{\psi}}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\delta\hat{\psi}}{\partial x^2} - 2gN|\Psi_0|^2\delta\hat{\psi} - gN\Psi_0^2\delta\hat{\psi}^\dagger. \tag{2}$$

Applying this to NLSE breathers, we use Gordon's solution of the NLSE [56] for two solitons with numbers of atoms N_1 and N_2, which contains eight free parameters (see the Supplemental Material [57] for derivation details, which includes references to relevant papers [58–61]). Four parameters represent the bosonic state as a whole: the total number of atoms $N = N_1 + N_2$, overall phase Θ, COM velocity V, and COM coordinate B. The other four parameters are the relative velocity v of the constituent solitons, initial distance between them, b, initial phase difference, θ, and mass difference, $n = N_2 - N_1$. The particular case of $n = \pm N/2$ and $v = b = \theta = 0$ corresponds to the breather solution. In the COM reference frame ($V = 0$), the breather remains localized, oscillating with period $T_{\rm br} = 32\pi\hbar^3/(mg^2N^2)$. On the other hand, the fundamental soliton is obtained for $n = \pm N$ and $v = b = \theta = 0$.

TABLE I. Initial values of the quantum fluctuations $\langle\Delta\hat{\chi}_0^2\rangle$ of the overall and relative parameters of the breather, obtained for the white-noise vacuum state.

	Number	Phase	Velocity	Coordinate
Overall	N	$(105+11\pi^2)/(315N)$	$N\bar{v}^2/192$	$16\pi^2\bar{x}^2/(3N^3)$
Relative	$N/5$	$4(420+23\pi^2)/(315N)$	$23N\bar{v}^2/420$	$256\pi^2\bar{x}^2/(15N^3)$

The quantum correction to the two-soliton solution is

$$\delta\hat{\psi} = \sum_\chi f_\chi(x,t)\Delta\hat{\chi}_0 + \hat{\psi}_{\rm cont}(x,t), \tag{3}$$

where χ is set of the eight parameters (N, Θ, V, B, n, θ, v, and b), and $f_\chi(x,t) = \partial(\sqrt{N}\Psi_0)/\partial\chi$ are derivatives of the MF solution with respect to them. Then, the sum in Eq. (3) is an exact operator solution of the linearized equation (2). Hermitian operators $\Delta\hat{\chi}_0$, introduced in Refs. [48,49], are considered as quantum fluctuations of the parameters at $t=0$, as they have the same effect on the density as classical fluctuations of the MF parameters, see [57]. The set of eight parameters is related to breaking of the $U(1)$ and translational symmetries of the underlying Hamiltonian, hence they represent the Goldstone and "lost" modes, in the framework of the Bogoliubov–de Gennes description [36,62,63]. The operator term $\hat{\psi}_{\rm cont}$ in Eq. (3) represents fluctuations with a continuum spectrum (which were analyzed for the fundamental soliton in Ref. [64]). In this work, we assume orthogonality of the continuum fluctuations $\hat{\psi}_{\rm cont}$ to the discrete-expansion modes, leaving a rigorous proof of this fact for subsequent work. Indeed, there are good reasons for this conjecture: first, in the context of nonlinear optics [49,51] it is supported by the fact that, in the limit of $t\to\infty$, continuum modes completely disperse out, hence the orthogonality condition definitely holds. Second, the orthogonality of the Goldstone and continuum modes is built into the procedure of the construction of Bogoliubov eigenstates [65].

Operators $\delta\hat{\psi}^\dagger$ and $\delta\hat{\psi}$ may be interpreted as creation and annihilation operators, respectively, of the quantum fluctuations. To properly define the action of the operators, one has to specify the nature of the vacuum state. The breather is initialized as a mother soliton, which defines the vacuum state of the quantum-fluctuation operators around the breather. Below we address two different physically relevant schemes for incorporating the vacuum state into the Bogoliubov method.

The white-noise vacuum.—The most common approach to introduce the vacuum state for the $\delta\hat{\psi}^\dagger$ and $\delta\hat{\psi}$ operators (in particular, in optics [6,49–51,64]) is to consider one with fluctuations in the form of uncorrelated random noise. Such a formulation is also adopted in atomic physics [42], and has the following interpretation: the mother soliton is a Hartree product of noninteracting single-particle wave functions, all having the shape of the mother soliton. Thus, only the product $\langle\delta\hat{\psi}(x,0)\delta\hat{\psi}^\dagger(x',0)\rangle = \delta(x-x')$, where the averaging $\langle\ldots\rangle$ is taken over the vacuum state, yields nonzero correlations (see [57]). At $t=0$, quantum fluctuations of eight parameters, $\Delta\hat{\chi}_0$, can be expressed in terms of overlaps of functions $f_\chi(x,t)$ as

$$\langle\Delta\hat{\chi}_0^2\rangle \propto \int_{-\infty}^{+\infty} dx|f_{\tilde{\chi}}(x,0)|^2 \tag{4}$$

(see [57]), with eight parameters combined in four pairs $(\chi,\tilde{\chi})$, *viz.*, (N,Θ), (V,B), (n,θ), and (v,b). The relationships between them resemble canonical conjugation (up to constant factors). (See [57] for details.) Derivatives of Gordon's solution and the overlap integrals were evaluated analytically (using Wolfram Mathematica). The so evaluated initial values of the fluctuations are presented in Table I, where scales of the length and velocity are $\bar{x} = \hbar^2/(mg)$ and $\bar{v} = g/\hbar$. For ^{7}Li atoms with $m = 7$ AMU, $\omega_\perp = 254\times 2\pi$Hz, and $a_{\rm sc} = -4a_0$ (a_0 is the Bohr radius), we have $\bar{x}\approx 1.34$ cm and $\bar{v}\approx 6.75\times 10^{-5}$ cm/s, while the breather's oscillation period is $T_{\rm br}\approx 4\times 10^6/N^2$ s.

Next, we compare these uncertainty expressions with the standard (Heisenberg's) ones:

$$\langle\Delta\hat{N}_0^2\rangle\langle\Delta\hat{\Theta}_0^2\rangle \approx 0.678 > 0.25 \tag{5a}$$

$$N^2m^2\langle\Delta\hat{V}_0^2\rangle\langle\Delta\hat{B}_0^2\rangle/\hbar^2 \approx 0.274 > 0.25, \tag{5b}$$

$$\langle\Delta\hat{\theta}_0^2\rangle\langle\Delta\hat{n}_0^2\rangle/4 \approx 0.41 > 0.25, \tag{5c}$$

$$N^2(3m/16)^2\langle\Delta\hat{v}_0^2\rangle\langle\Delta\hat{b}_0^2\rangle/\hbar^2 \approx 0.3243 > 0.25, \tag{5d}$$

where the rightmost bound comes from the exact commutation relations between $-\Theta$, B, $-\theta$, and b, and the corresponding "momenta" $\hbar N$, NmV, $\hbar n/2$, and $3Nmv/16$, respectively, see [57].

Note that the uncertainty value for the relative momentum, $3mNv/16$, and distance, b, is ≈20% larger than that for COM momentum-position pair. One can also evaluate averages of the cross products of the operators, using formulas similar to Eq. (4), see [57]. Nonvanishing values

$$\langle\Delta\hat{N}_0\Delta\hat{\Theta}_0\rangle = i/2, \qquad \langle\Delta\hat{B}_0\Delta\hat{V}_0\rangle = i\hbar/(2Nm),$$
$$\langle\Delta\hat{n}_0\Delta\hat{\theta}_0\rangle = i, \qquad \langle\Delta\hat{b}_0\Delta\hat{v}_0\rangle = 8i\hbar/(3Nm), \tag{6}$$

are purely imaginary due to properties of modes $f_{\tilde{\chi}}$, and $\langle\Delta\hat{\chi}_0\Delta\hat{\chi}_0'\rangle = -\langle\Delta\hat{\chi}_0'\Delta\hat{\chi}_0\rangle$ due to the hermiticity. Note that

$\sqrt{\langle\Delta\hat{N}_0^2\rangle\langle\Delta\hat{\Theta}_0^2\rangle}\approx 0.82$, $\sqrt{\langle\Delta\hat{V}_0^2\rangle\langle\Delta\hat{B}_0^2\rangle}\approx 2.1\hbar/(Nm)$, $\sqrt{\langle\Delta\hat{n}_0^2\rangle\langle\Delta\hat{\theta}_0^2\rangle}\approx 1.3$, and $\sqrt{\langle\Delta\hat{v}_0^2\rangle\langle\Delta\hat{b}_0^2\rangle}\approx 3\hbar/(Nm)$. Then, the cross term $\langle\Delta\hat{B}_0\Delta\hat{V}_0\rangle$ may be neglected, while others are non-negligible.

Contributions from mother-soliton's continuum fluctuations.—The predictions for fluctuations of the breather's parameters are significantly different if field fluctuations of the mother (prequench) soliton are included. In contrast to the white-noise vacuum case, we cannot keep only one product of the fluctuating operators, $\delta\hat{\psi}(x)\delta\hat{\psi}^\dagger(x')$, therefore the correlated-quantum-noise vacuum leads to different expectation values. In turn, quantum fluctuations of the fundamental mother soliton can be separated into discrete and continuum parts [36,62,63]. Further, expectation values of the continuum creation or annihilation operator products can be calculated using known exact expressions [36,66] for the Bogoliubov modes of the fundamental soliton (see [57]). Fluctuations of discrete parameters of the mother soliton are determined by derivatives of the mean field with respect to these parameters. They coincide with the breather's overall (COM) fluctuations, as the soliton's and breather's mean fields are the same at $t=0$. Because fluctuations of the discrete parameters of the mother soliton are decoupled from the relative degrees of freedom of the breather, they do not affect the corresponding variances. Uncertainties of the overall degrees of freedom of the breather are determined by parameters of the experiment that creates the mother soliton. Note also that, due to phase-diffusion effects [62,63], fluctuations of the discrete parameters of the mother soliton depend on time between the creation of the soliton and the application of the interaction quench to it.

In Table II we compare initial variances of the relative parameters for different vacuum states. Due to the complicated form of the expressions, the variances for the correlated-noise vacuum were evaluated numerically. The difference, while not being enormous, is evident and it may manifest itself in the observable dynamics of the breather. The cross-product averages are the same as for the white-noise vacuum, see Eq. (6).

The BA estimates for the relative velocity variance obtained for small N [52], $0.035N\bar{v}^2$, is within 20% of the correlated-vacuum prediction. This conclusion is a significant result of the present work, as it demonstrates that the crucially important characteristics of the fluctuational dynamics are close for different vacuum states and in the opposite limits of small and large N, thus revealing universal features of the dynamics, which should be amenable to experimental observation.

In Fig. 2 we display the evolution of the variances of quantum operators of the relative parameters of the

TABLE II. Initial quantum fluctuations of relative parameters of the breather for the white-noise and prequench correlated-vacuum states.

Noise	$\langle\Delta\hat{n}_0^2\rangle$	$\langle\Delta\hat{\theta}_0^2\rangle$	$\langle\Delta\hat{v}_0^2\rangle$	$\langle\Delta\hat{b}_0^2\rangle$
White	$0.2N$	$8.22/N$	$0.0548N\bar{v}^2$	$168\bar{x}^2/N^3$
Correlated	$0.3N$	$6.26/N$	$0.0429N\bar{v}^2$	$198\bar{x}^2/N^3$

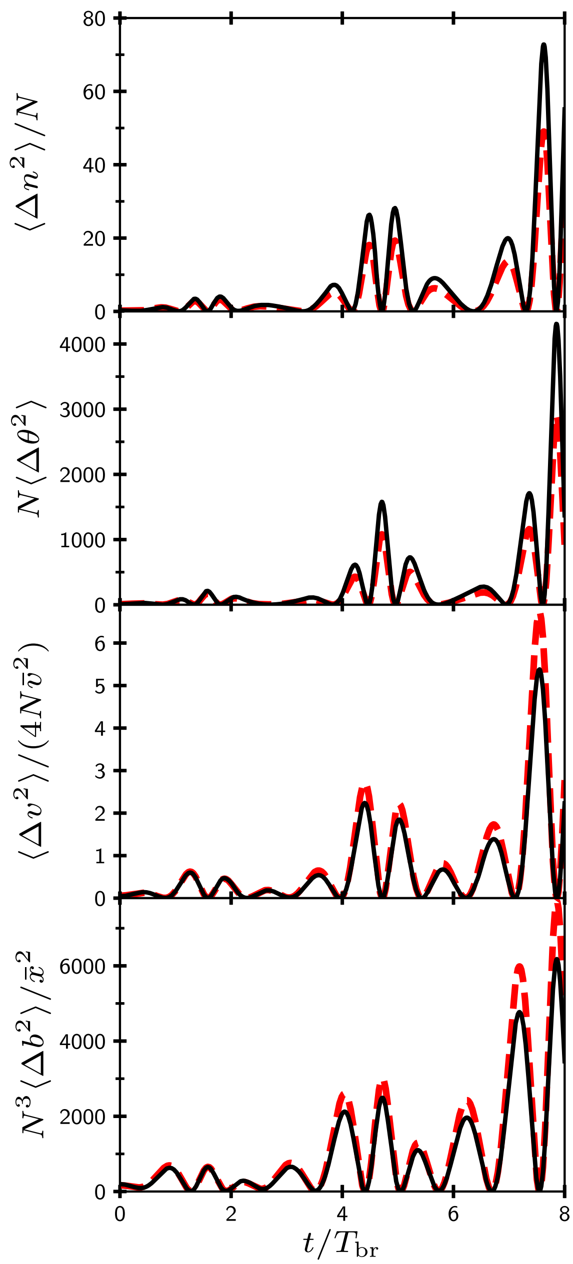

FIG. 2. Variances of fluctuations of the relative parameters of the breather, as a function of time (from top to bottom): the number of atoms $\langle\Delta\hat{n}^2(t)\rangle$, phase $\langle\Delta\hat{\theta}^2(t)\rangle$, velocity $\langle\Delta\hat{v}^2(t)\rangle$, and position $\langle\Delta\hat{b}^2(t)\rangle$, as found for the white-noise vacuum state (red dashed lines) and prequench correlated vacuum state (black solid lines).

breather, and compare the results for the white-noise and prequench correlated-noise vacuum states.

The splitting of the breather can be detected once the constituent solitons are separated by a distance comparable to the breather's width, which is $l_{\rm br} = 8\hbar^2/(mgN) \approx 36\ \mu$m [28] under realistic experimental conditions ($N = 3 \times 10^3$ ^{7}Li atoms with the parameters mentioned above). Therefore, the time of dissociation due to quantum fluctuations, $\tau = l_{\rm br}/\sqrt{\langle \Delta v_0^2 \rangle}$, depends on the vacuum state (see Table II), namely, $\tau_{\rm white} \approx 4.16$ s, and $\tau_{\rm corr} \approx 4.7$ s. Thus, the inclusion of the continuum fluctuations of the mother soliton increases the dissociation time by more than a breather's period (≈0.22 s). Note that the BA estimate for small N [52], ≈3 s, used a different technical definition of the dissociation time; using the present definition, the BA yields $\tau_{\rm BA} \approx 5.18$ s. Eventually, the results again clearly corroborate the inference that the fluctuational dynamics reveals universal features, amenable to experimental observation, in both limits of small and large N.

As noted above, the spontaneous dissociation is forbidden in the integrable 1D axially uniform MF model. The integrability maintains robustness of solitons and "debris," such as radiation or additional small-amplitude solitons, created in the experiment. The "debris" cannot bind into the breather, and would disperse by themselves. In principle, dissociation may be induced by integrability-breaking 3D effects, decoherence, or an axial potential, which all are unavoidable in the experiment. However, 3D MF calculations [67], as well as analytical and numerical analyses [68–71] of the decoherence, induced by the linear loss, do not reveal any dissociation. Besides, the relative motion of the constituent solitons is rather insensitive to long-scale potentials since linear potentials depend only on the COM coordinate and the quadratic ones cannot induce dissociation due to parity conservation. Calculations [47] demonstrated dissociation due to a narrow potential barrier (of the width $\delta x \ll l_{\rm br}$) above certain threshold, namely, the potential δU does not induce dissociation if $\delta U \delta x \lesssim 10^{-4} gN$. This condition is not too strong in real experiments with large N. Thus, the dissociation into daughter solitons can only be a result of quantum noise.

Conclusions.—The Bogoliubov linearization approach makes it possible to estimate variances of the quantum fluctuations of the breather's discrete parameters, including its COM and relative degrees of freedom. We consider two cases of the vacuum state: an easier, tractable uncorrelated quantum noise, alias "white noise," and a state with the correlated quantum noise, that takes into account quantum fluctuations of the mother soliton. The comparison shows that the correlated noise noticeably changes initial values and the evolution of the variances. Disagreement between the relative-velocity variance for the correlated noise and BA results [52], obtained for the small number of atoms, $N \leq 23$, is $< 20\%$. The present analysis produces variances of other breather parameters as well. A fundamental observable effect that quantum fluctuations can induce is dissociation of the breather. This effect is essentially the same, irrespective of the choice of the noise pattern. Namely, the dissociation time, estimated for realistic experimental parameters as $\tau_{\rm corr} \approx 4.7$ s for the correlated-noise vacuum, is about one breather period larger than for the uncorrelated noise and closer to the BA estimate ≈5.18 s for small N. The proximity of the basic results obtained for small and large N and for different vacuum states is a strong indication that the quantum dynamics of breathers is dominated by the universal features. Thus, the results reveal the feasibility of the observation of direct manifestations of quantum fluctuations in macroscopic degrees of freedom—in particular, the relative velocity of the two initially bound solitons. Note also that the proximity of the uncertainty relation (5d) to the lower limit of the Heisenberg's position-momentum uncertainty relation indicates that the state of the relative motion is *probably* a macroscopically quantum one: if it were spread over a large phase-space area, it would—while remaining formally a pure state—become chaotic in the course of the subsequent quantum evolution, while we see that it does not do that.

This work was jointly supported by the National Science Foundation through Grants No. PHY-1402249, No. PHY-1607221, and No. PHY-1912542 and the Binational (U.S.-Israel) Science Foundation through Grant No. 2015616. The work at Rice was supported by the Army Research Office Multidisciplinary University Research Initiative (Grant No. W911NF-14-1-0003), the Office of Naval Research, the NSF (Grant No. PHY-1707992), and the Welch Foundation (Grant No. C-1133). The work at T. A. U. was supported by the Israel Science Foundation (Grant No. 1286/17) and by the Tel Aviv University—Swinburne University of Technology Research Collaboration award. O. V. M. acknowledges the support from the German Aeronautics and Space Administration (DLR) through Grant No. 50 WM 1957.

*oleksandr.marchukov@tu-darmstadt.de

[1] B. D. Fried and Y. H. Ichikawa, On the nonlinear Schrödinger equation for Langmuir waves, J. Phys. Soc. Jpn. **34**, 1073 (1973).
[2] A. Hasegawa and F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion, Appl. Phys. Lett. **23**, 142 (1973).
[3] A. Hasegawa and F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. II. Normal dispersion, Appl. Phys. Lett. **23**, 171 (1973).
[4] Y. Lai and H. A. Haus, Quantum theory of solitons in optical fibers. I. Time-dependent Hartree approximation, Phys. Rev. A **40**, 844 (1989).
[5] Y. Lai and H. A. Haus, Quantum theory of solitons in optical fibers. II. Exact solution, Phys. Rev. A **40**, 854 (1989).

[6] H. A. Haus and W. S. Wong, Solitons in optical communications, Rev. Mod. Phys. **68**, 423 (1996).
[7] K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, Formation and propagation of matter-wave soliton trains, Nature (London) **417**, 150 (2002).
[8] L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, Formation of a matter-wave bright soliton, Science **296**, 1290 (2002).
[9] K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, Bright matter wave solitons in Bose-Einstein condensates, New J. Phys. **5**, 73 (2003).
[10] S. L. Cornish, S. T. Thompson, and C. E. Wieman, Formation of Bright Matter-Wave Solitons During the Collapse of Attractive Bose-Einstein Condensates, Phys. Rev. Lett. **96**, 170401 (2006).
[11] A. L. Marchant, T. P. Billam, T. P. Wiles, M. M. Yu, S. A. Gardiner, and S. L. Cornish, Controlled formation and reflection of a bright solitary matter-wave, Nat. Commun. **4**, 1865 (2013).
[12] N. Bogolubov, On the theory of superfluidity, J. Phys. (USSR) **XI**, 23 (1947).
[13] *Bose-Einstein Condensation*, edited by A. Griffin, D. W. Snoke, and S. Stringari (Cambridge University Press, Cambridge, England, 1995).
[14] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Theory of Bose-Einstein condensation in trapped gases, Rev. Mod. Phys. **71**, 463 (1999).
[15] A. J. Leggett, Bose-Einstein condensation in the alkali gases: Some fundamental concepts, Rev. Mod. Phys. **73**, 307 (2001).
[16] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge, England, 2008), Vol. 9780521846, pp. 1–569.
[17] T. D. Lee, K. Huang, and C. N. Yang, Eigenvalues and eigenfunctions of a Bose system of hard spheres and its low-temperature properties, Phys. Rev. **106**, 1135 (1957).
[18] D. S. Petrov, Quantum Mechanical Stabilization of a Collapsing Bose-Bose Mixture, Phys. Rev. Lett. **115**, 155302 (2015).
[19] D. S. Petrov and G. E. Astrakharchik, Ultradilute Low-Dimensional Liquids, Phys. Rev. Lett. **117**, 100401 (2016).
[20] Y. Li, Z. Chen, Z. Luo, C. Huang, H. Tan, W. Pang, and B. A. Malomed, Two-dimensional vortex quantum droplets, Phys. Rev. A **98**, 063602 (2018).
[21] Y. V. Kartashov, B. A. Malomed, L. Tarruell, and L. Torner, Three-dimensional droplets of swirling superfluids, Phys. Rev. A **98**, 013612 (2018).
[22] C. R. Cabrera, L. Tanzi, J. Sanz, B. Naylor, P. Thomas, P. Cheiney, and L. Tarruell, Quantum liquid droplets in a mixture of Bose-Einstein condensates, Science **359**, 301 (2018).
[23] P. Cheiney, C. R. Cabrera, J. Sanz, B. Naylor, L. Tanzi, and L. Tarruell, Bright Soliton to Quantum Droplet Transition in a Mixture of Bose-Einstein Condensates, Phys. Rev. Lett. **120**, 135301 (2018).
[24] G. Semeghini, G. Ferioli, L. Masi, C. Mazzinghi, L. Wolswijk, F. Minardi, M. Modugno, G. Modugno, M. Inguscio, and M. Fattori, Self-Bound Quantum Droplets of Atomic Mixtures in Free Space, Phys. Rev. Lett. **120**, 235301 (2018).
[25] C. D'Errico, A. Burchianti, M. Prevedelli, L. Salasnich, F. Ancilotto, M. Modugno, F. Minardi, and C. Fort, Observation of quantum droplets in a heteronuclear bosonic mixture, Phys. Rev. Research **1**, 033155 (2019).
[26] M. Schmitt, M. Wenzel, F. Böttcher, I. Ferrier-Barbut, and T. Pfau, Self-bound droplets of a dilute magnetic quantum liquid, Nature (London) **539**, 259 (2016).
[27] I. Ferrier-Barbut, H. Kadau, M. Schmitt, M. Wenzel, and T. Pfau, Observation of Quantum Droplets in a Strongly Dipolar Bose Gas, Phys. Rev. Lett. **116**, 215301 (2016).
[28] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP **34**, 62 (1972), http://www.jetp.ac.ru/cgi-bin/e/index/e/34/1/p62?a=list.
[29] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (Society for Industrial and Applied Mathematics, Philadelphia, 1981).
[30] S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method (Monographs in Contemporary Mathematics)* (Consultants Bureau, New York, 1984), p. 292.
[31] J. Satsuma and N. Yajima, Initial value problems of one-dimensional self-modulation of nonlinear waves in dispersive media, Prog. Theor. Phys. Suppl. **55**, 284 (1974).
[32] Y. S. Kivshar and B. A. Malomed, Dynamics of solitons in nearly integrable systems, Rev. Mod. Phys. **61**, 763 (1989).
[33] A. Di Carli, C. D. Colquhoun, G. Henderson, S. Flannigan, G.-L. Oppo, A. J. Daley, S. Kuhr, and E. Haller, Excitation Modes of Bright Matter-Wave Solitons, Phys. Rev. Lett. **123**, 123602 (2019).
[34] P. D. Drummond and W. Man, Quantum noise in reversible soliton logic, Opt. Commun. **105**, 99 (1994).
[35] T. Vaughan, P. Drummond, and G. Leuchs, Quantum limits to center-of-mass measurements, Phys. Rev. A **75**, 033617 (2007).
[36] Y. Castin, Internal structure of a quantum soliton and classical excitations due to trap opening, Eur. Phys. J. B **68**, 317 (2009).
[37] A. Boissé, G. Berthet, L. Fouché, G. Salomon, A. Aspect, S. Lepoutre, and T. Bourdel, Nonlinear scattering of atomic bright solitons in disorder, Europhys. Lett. **117**, 10007 (2017).
[38] E. Schmidt, L. Knöll, D.-G. Welsch, M. Zielonka, F. König, and A. Sizmann, Enhanced Quantum Correlations in Bound Higher-Order Solitons, Phys. Rev. Lett. **85**, 3801 (2000).
[39] E. Schmidt, L. Knöll, and D.-G. Welsch, Quantum noise of damped n-solitons, Opt. Commun. **179**, 603 (2000).
[40] R.-K. Lee, Y. Lai, and Y. S. Kivshar, Quantum correlations in soliton collisions, Phys. Rev. A **71**, 035801 (2005).
[41] P. D. Drummond and B. Opanchuk, Truncated Wigner dynamics and conservation laws, Phys. Rev. A **96**, 043616 (2017).
[42] B. Opanchuk and P. D. Drummond, One-dimensional Bose gas dynamics: Breather relaxation, Phys. Rev. A **96**, 053628 (2017).
[43] K. L. Ng, B. Opanchuk, M. D. Reid, and P. D. Drummond, Nonlocal Pair Correlations in a Higher-Order Bose Gas Soliton, Phys. Rev. Lett. **122**, 203604 (2019).
[44] S.-W. Su, S.-C. Gou, I. Kang Liu, A. S. Bradley, O. Fialko, and J. Brand, Oscillons in coupled Bose-Einstein condensates, Phys. Rev. A **91**, 023631 (2015).

[45] D. Witthaut, F. Trimborn, H. Hennig, G. Kordas, T. Geisel, and S. Wimberger, Beyond mean-field dynamics in open Bose-Hubbard chains, Phys. Rev. A **83**, 063608 (2011).

[46] H. Yanay, L. Khaykovich, and B. A. Malomed, Stabilization and destabilization of second-order solitons against perturbations in the nonlinear Schrödinger equation, Chaos **19**, 033145 (2009).

[47] O. V. Marchukov, B. A. Malomed, V. A. Yurovsky, M. Olshanii, V. Dunjko, and R. G. Hulet, Splitting of nonlinear-Schrödinger-equation breathers by linear and nonlinear localized potentials, Phys. Rev. A **99**, 063623 (2019).

[48] H. A. Haus and M. N. Islam, Theory of the soliton laser, IEEE J. Quantum Electron. **21**, 1172 (1985).

[49] H. A. Haus and Y. Lai, Quantum theory of soliton squeezing: A linearized approach, J. Opt. Soc. Am B **7**, 386 (1990).

[50] Y. Lai, Quantum theory of soliton propagation: A unified approach based on the linearization approximation, J. Opt. Soc. Am. B **10**, 475 (1993).

[51] C.-P. Yeang, Quantum theory of a second-order soliton based on a linearization approximation, J. Opt. Soc. Am. B **16**, 1269 (1999).

[52] V. A. Yurovsky, B. A. Malomed, R. G. Hulet, and M. Olshanii, Dissociation of One-Dimensional Matter-Wave Breathers due to Quantum Many-Body Effects, Phys. Rev. Lett. **119**, 220401 (2017).

[53] V. A. Yurovsky, M. Olshanii, and D. S. Weiss, Collisions, correlations, and integrability in atom waveguides, Adv. At. Mol. Opt. Phys. **55**, 61 (2008).

[54] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Feshbach resonances in ultracold gases, Rev. Mod. Phys. **82**, 1225 (2010).

[55] M. Olshanii, Atomic Scattering in the Presence of an External Confinement and a Gas of Impenetrable Bosons, Phys. Rev. Lett. **81**, 938 (1998).

[56] J. P. Gordon, Interaction forces among solitons in optical fibers, Opt. Lett. **8**, 596 (1983).

[57] See the Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.125.050405 for derivation details.

[58] A. Griffin, Conserving and gapless approximations for an inhomogeneous Bose gas at finite temperatures, Phys. Rev. B **53**, 9341 (1996).

[59] J. Andersen, Theory of the weakly interacting Bose gas, Rev. Mod. Phys. **76**, 599 (2004).

[60] L. D. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer-Verlag, Berlin, 2007), reprint of the 1987 edition.

[61] L. Landau and E. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Pergamon Press, New York, 1977).

[62] M. Lewenstein and L. You, Quantum Phase Diffusion of a Bose-Einstein Condensate, Phys. Rev. Lett. **77**, 3489 (1996).

[63] Y. Castin and R. Dum, Low-temperature Bose-Einstein condensates in time-dependent traps: Beyond the $U(1)$ symmetry-breaking approach, Phys. Rev. A **57**, 3008 (1998).

[64] H. A. Haus and C. X. Yu, Soliton squeezing and the continuum, J. Opt. Soc. Am. B **17**, 618 (2000).

[65] D. A. Takahashi and M. Nitta, Counting rule of Nambu-Goldstone modes for internal and spacetime symmetries: Bogoliubov theory approach, Ann. Phys. (Amsterdam) **354**, 101 (2015).

[66] D. J. Kaup, Perturbation theory for solitons in optical fibers, Phys. Rev. A **42**, 5689 (1990).

[67] J. Golde, J. Ruhl, M. Olshanii, V. Dunjko, S. Datta, and B. A. Malomed, Metastability versus collapse following a quench in attractive Bose-Einstein condensates, Phys. Rev. A **97**, 053604 (2018).

[68] N. R. Pereira and F. Y. F. Chu, Damped double solitons in the nonlinear Schrödinger equation, Phys. Fluids **22**, 874 (1979).

[69] K. Blow and N. Doran, The asymptotic dispersion of soliton pulses in lossy fibres, Opt. Commun. **52**, 367 (1985).

[70] B. A. Malomed, Inelastic interactions of solitons in nearly integrable systems. I, Physica (Amsterdam) **15D**, 374 (1985).

[71] E. M. Dianov, Z. S. Nikonova, and V. N. Serkin, Influence of optical losses on the dynamics of nonlinear propagation of pulses in single-mode fiber waveguides, Sov. J. Quantum Electron. **16**, 219 (1986).